Generalized Uncertainty Relation in Thermodynamics

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Abstract

A generalization of the thermodynamic uncertainty relations is proposed. It is done by introducing of an additional term proportional to the interior energy into the standard thermodynamic uncertainty relation that leads to existence of the lower limit of inverse temperature.

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It is well known that in thermodynamics an inequality for the pair interior energy - inverse temperature, which is completely analogous to the standard uncertainty relation in quantum mechanics [1] can be written down [2] – [4]. The only (but essential) difference of this inequality from the quantum mechanical one is that the main quadratic fluctuation is defined by means of classical partition function rather than by quantum mechanical expectation values. In the last 14 - 15 years a lot of papers appeared in which the usual momentum-coordinate uncertainty relation has been modified at very high energies of order Planck energy E_p [5]–[9]. In this note we propose simple reasons for modifying the thermodynamic uncertainty relation at Planck energies. This modification results in existence of the minimal possible main quadratic fluctuation of the inverse

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temperature. Of course we assume that all the thermodynamic quantities used are properly defined so that they have physical sense at such high energies.

We start with usual Heisenberg uncertainty relations [1] for momentum - coordinate:

$$\Delta x \geq \frac{\hbar}{\Delta p}. \tag{1}$$

It was shown that at the Planck scale a high-energy term must appear:

$$\Delta x \geq \frac{\hbar}{\Delta p} + const\, L_p^2\, \frac{\Delta p}{\hbar}. \tag{2}$$

where L_p is the Planck length $L_p^2 = G\hbar/c^3 \simeq 1,6\; 10^{-35}m$. In [5] this term is derived from the string theory, in [6] it follows from the simple estimates of Newtonian gravity and quantum mechanics, in [7] it comes from the black hole physics, other methods can also be used [8],[9]. Particularly the coefficient *const* is shown to be unity in paper [6]. Relation (2) is quadratic in Δp

$$L_p^2\,(\Delta p)^2 - \hbar\,\Delta x \Delta p + \hbar^2 \leq 0 \tag{3}$$

and therefore leads to the fundamental length

$$\Delta x_{min} = 2L_p \tag{4}$$

Using relations (2) it is easy to obtain a similar relation for the energy - time pair. Indeed (2) gives

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + L_p^2\, \frac{\Delta p}{c\hbar}, \tag{5}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \frac{L_p^2}{c^2}\,\frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + t_p^2\,\frac{\Delta E}{\hbar}. \tag{6}$$

where the smallness of L_p is taken into account so that the difference between ΔE and $\Delta(pc)$ can be neglected and t_p is the Planck time $t_p = L_p/c = \sqrt{G\hbar/c^5} \simeq 0,54\; 10^{-43}sec$. Inequality (6) gives analogously to (2) the lower boundary for time $\Delta t \geq 2t_p$ determining the fundamental time

$$\Delta t_{min} = 2t_p. \tag{7}$$

Thus, the inequalities discussed can be rewritten in a standard form

$$\left\{ \begin{array}{ll} \Delta x & \geq \dfrac{\hbar}{\Delta p} + \left(\dfrac{\Delta p}{P_{pl}}\right) \dfrac{\hbar}{P_{pl}} \\ \\ \Delta t & \geq \dfrac{\hbar}{\Delta E} + \left(\dfrac{\Delta E}{E_p}\right) \dfrac{\hbar}{E_p} \end{array} \right. \tag{8}$$

where $P_{pl} = E_p/c = \sqrt{\hbar c^3/G}$. Now we consider the thermodynamics uncertainty relations between the inverse temperature and interior energy of a macroscopic ensemble

$$\Delta \frac{1}{T} \geq \frac{k}{\Delta U}. \tag{9}$$

where k is the Boltzmann constant.

N.Bohr [2] and W.Heisenberg [3] first pointed out that such kind of uncertainty principle should take place in thermodynamics. The thermodynamic uncertainty relations (9) were proved by many authors and in various ways [4]. Therefore their validity does not raise any doubts. Nevertheless, relation (9) was proved in view of the standard model of the infinite-capacity heat bath encompassing the ensemble. But it is obvious from the above inequalities that at very high energies the capacity of the heat bath can no longer to be assumed infinite at the Planck scale. Indeed, the total energy of the pair heat bath - ensemble may be arbitrary large but finite merely as the universe is born at a finite energy. Hence the quantity that can be interpreted as the temperature of the ensemble must have the upper limit and so does its main quadratic deviation. In other words the quantity $\Delta(1/T)$ must be bounded from below. But in this case an additional term should be introduced into (9)

$$\Delta \frac{1}{T} \geq \frac{k}{\Delta U} + \eta\, \Delta U \tag{10}$$

where η is a coefficient. Dimension and symmetry reasons give

$$\eta = \frac{k}{E_p^2}.$$

As in the previous cases inequality (10) leads to the fundamental (inverse) temperature.

$$T_{max} = \frac{\hbar}{2t_p k} = \frac{\hbar}{\Delta t_{min} k}, \quad \beta_{min} = \frac{1}{kT_{max}} = \frac{\Delta t_{min}}{\hbar} \tag{11}$$

Thus, we obtain the system of generalized uncertainty relations in a symmetric form

$$\left\{\begin{array}{lcl} \Delta x & \geq & \frac{\hbar}{\Delta p} + \left(\frac{\Delta p}{P_{pl}}\right) \frac{\hbar}{P_{pl}} \\ \\ \Delta t & \geq & \frac{\hbar}{\Delta E} + \left(\frac{\Delta E}{E_p}\right) \frac{\hbar}{E_p} \\ \\ \Delta \frac{1}{T} & \geq & \frac{k}{\Delta U} + \left(\frac{\Delta U}{E_p}\right) \frac{k}{E_p} \end{array}\right. \qquad (12)$$

or in the equivalent form

$$\left\{\begin{array}{lcl} \Delta x & \geq & \frac{\hbar}{\Delta p} + L_p^2 \frac{\Delta p}{\hbar} \\ \\ \Delta t & \geq & \frac{\hbar}{\Delta E} + t_p^2 \frac{\Delta E}{\hbar} \\ \\ \Delta \frac{1}{T} & \geq & \frac{k}{\Delta U} + \frac{1}{T_p^2} \frac{\Delta U}{k} \end{array}\right. \qquad (13)$$

Here T_p is the Planck temperature: $T_p = \frac{E_p}{k}$.
In the conclusion we would like to note that the restriction on the heat bath made above turns the equilibric partition function to be non-Gibbsian [10].
After the issue of our principal work [11] devoted to the unification of the generalized uncertainty relations in quantum theory and thermodynamics, Carlos Castro has published an electronic preprint [12], where with a reference to our work he is attempting at justification of our primary result proceeding from more common considerations. However, he resorts to substitution, and we are of the opinion that:

$$t \mapsto \frac{1}{T}, \hbar \mapsto k, \triangle E \mapsto \triangle U$$

could not be accepted as a rigorous proof for the primary result of our paper. There is a reason to believe that a rigorous justification for the last (thermodynamic) inequalities in systems (12) and (13) may be made by the way of the deformation of Gibbs distribution.
Let us outline the main aspects of above-considered deformation. In our

opinion it could be obtained as the result of density-matrix deformation in Statistical Mechanics (see [13], Section 2, Paragraph 3):

$$\rho = \sum_n \omega_n |\varphi_n >< \varphi_n|, \tag{14}$$

where probability is given by

$$\omega_n = \frac{1}{Q} \exp(-\beta E_n).$$

Deformation of density matrix ρ (14) can be carried out similarly to deformation of density matrix (density pro-matrix) in Quantum Mechanics at Planck's scale (see [14],[15]). Proceeding with this analogy density matrix ρ in (14) should be changed by $\rho(\tau)$, where τ is a parameter of deformation. Deformed density matrix must fulfill the condition $\rho(\tau) \approx \rho$ when $T \ll T_p$. By analogy with [14],[15], only probabilities ω_n are subject of deformation in (14), changing by $\omega_n(\tau)$ and correspondingly deformed statistical density matrix is

$$\rho(\tau) = \sum_n \omega_n(\tau) |\varphi_n >< \varphi_n|. \tag{15}$$

This approach in our opinion could give us the possibility to obtain Deformed Canonical Distribution as well as a rigorous proof of thermodynamical general uncertainty relations.

An detailed analysis of deformation in Statistical Mechanics is an object of future investigations.

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